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Modulation of waves near the marginal state of instability in fluid-filled distensible tubes

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Abstract. One-dimensional wave propagation near the marginal state of modulational instability in an infinitely long, straight and homogeneous nonlinear elastic tube filled with an incompressible, inviscid fluid is considered. Using the reductive perturbation method, the amplitude modulation of weakly nonlinear waves is examined. It is shown that the amplitude modulation of these waves near the marginal state is governed by a generalized nonlinear Schrödinger equation (GNLS). Some exact solutions including oscillatory and solitary waves of the GNLS equation are presented.

1. Introduction

The nonlinear Schrödinger (NLS) equation is the simplest representative equation describing the self-modulation of one-dimensional monochromatic plane waves in dispersive media. The NLS equation is usually derived by balancing the nonlinearity and the width of the side-band of the quasi-monochromatic wave when both effects are assumed to be small but finite of the order ϵ. The NLS equation has a plane-wave solution of constant amplitude; the modulational instability of this plane wave is determined by the sign of the product of the coefficients of the nonlinear and dispersive terms of the NLS equation. The plane-wave solution to the NLS equation is modulationally unstable if the sign of the product is positive and modulationally stable if the sign of the product is negative. Therefore, the marginal state of the modulational instability is given by the condition that the coefficient of the nonlinear term or the coefficient of the dispersive term vanishes. In such a case, i.e. at the marginal state, the NLS equation is no longer valid. In other words, the asymptotic expansion used to obtain the NLS equation is not valid in the vicinity of the critical wavenumber. Thus, to take into account nonlinearity, the asymptotic expansion should be modified and the effect of nonlinearity should be intensified; then the resulting evolution equation may include higher order nonlinear terms compared to those in the NLS equation. The primary objective of this paper is to discuss such a case within the context of wave propagation in fluid-filled distensible tubes.

The problem of nonlinear self-modulation of small-but-finite amplitude waves in fluid-filled distensible tubes has been considered by several authors. Ravindran and Prasad [1] showed for a *linear elastic* tube-wall model that the nonlinear self-modulation of pressure waves is governed by a NLS equation. The same problem was investigated for a *nonlinear visco-elastic* tube-wall model by Erbay and Erbay [2]. It was shown that the amplitude modulation of pressure waves for a nonlinear visco-elastic tube filled with an incompressible, inviscid fluid can be described by a dissipative NLS equation. In the absence of dissipative effects this equation reduces to the classical NLS equation which corresponds to the case of a nonlinear elastic tube. The modulational instability of plane-wave solutions of the NLS equation was also studied in [2] for distensible tubes made of various elastic materials. It was observed that the plane-wave solution of the NLS equation is always modulationally stable for both linear elastic and Mooney–Rivlin elastic materials. However, for the other two materials, i.e. those proposed by Ishiara *et al* [3] and Demiray [4], this is not the case and the plane wave may be modulationally stable or unstable depending on initial deformation and wavenumber. Thus, it follows that for these two materials the NLS equation is no longer valid at the marginal state where the coefficient of the nonlinear term vanishes. In this paper, for fluid-filled nonlinear elastic tubes, the amplitude modulation of plane waves at a critical wavenumber for which the coefficient of the nonlinear term of the NLS equation vanishes is considered. Since the nonlinear term vanishes in the marginal state, one should introduce a new ordering to balance the nonlinearity and the band-width. Without changing the order of the band-width of quasi-monochromatic waves ($\mathcal{O}(\epsilon)$), the effect of nonlinearity is intensified by assuming the order of nonlinearity as $\mathcal{O}(\epsilon^{1/2})$. The reductive perturbation technique [5] is used to derive the appropriate modulation equation when the wavenumber is near to the critical wavenumber. Through the rescaling, higher-order nonlinear terms appear in the resulting evolution equation which is called the generalized nonlinear Schrödinger equation (GNLS).

The one-dimensional model in this paper was originally proposed by Tait and Moodie [6]; in order to incorporate the geometric dispersion into the model, the tube-wall inertia is added to the pressure–area relation in [7]. The governing equations of the fluid in the tube, and the pressure–area relation for the tube wall are summarized in section 2. The dimensionless forms of the governing equations are also given in this section. For comparison purposes the pressure–area relations corresponding to various incompressible elastic materials are presented.

Section 3 is devoted to the study of nonlinear modulation of small-but-finite amplitude waves near the marginal state of instability. First, the problem of amplitude modulation in fluid-filled nonlinear elastic tubes is briefly examined and the possibility of having marginal states for the NLS equation is discussed; it is pointed out that some elastic materials, i.e. those proposed by Ishiara *et al* and Demiray, have marginal states. In the same section, using the reductive perturbation method, the GNLS equation which is valid near the marginal state is derived.

In section 4, using the approach given in [8], the travelling wave solutions of the GNLS equation are presented. These solutions of the GNLS equation include both oscillatory and solitary waves.

2. Governing equations

We consider an infinitely long, straight and homogeneous tube filled with an incompressible, inviscid fluid. For a one-dimensional model, the governing equations for fluid flow are the conservation of mass and the momentum equations given by

$$\frac{\partial A}{\partial t}+\frac{\partial (Av)}{\partial z}=0 \tag{2.1}$$

and

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}+\frac{1}{\rho_f}\frac{\partial P}{\partial z}=0 \tag{2.2}$$

respectively, where A denotes the internal area of the tube, z is the axial distance, t is time, v is used to denote the fluid velocity in the z direction, ρ_f represents the density of the fluid, and P represents the pressure difference between the inside and outside of the tube. These equations involve three unknowns, namely $P(z,t)$, $A(z,t)$ and $v(z,t)$ and therefore, a third equation, the so-called 'pressure–area relation', is needed to complete the set of equations. The pressure–area relation used in this paper is derived in [6] for a nonlinear visco-elastic material, which includes the nonlinear elastic case as a special case, and is extended in [7] by adding the tube-wall inertia effects to incorporate the geometric dispersion into the model .

For a nonlinear elastic material the pressure–area relation may be written in the following form:

$$\frac{A}{A_0}P - \frac{\rho_s}{4\pi}\frac{H}{R_0}\left[\frac{\partial^2 A}{\partial t^2} - \frac{1}{A}\left(\frac{\partial A}{\partial t}\right)^2\right] = \mu\frac{H}{R_0}\Phi\left(\frac{A}{A_0}\right) \tag{2.3}$$

where ρ_s is the density of the wall material, μ is a material constant, and H, R_0 and A_0 are the thickness, the inner radius and the internal area of the tube in the undeformed state respectively. The function $\Phi(A/A_0)$ is determined by the elastic properties of the tube wall and vanishes when $A = A_0$. The explicit forms of $\Phi(A/A_0)$ corresponding to various elastic materials will be given below.

Now, introducing the following non-dimensional quantities:

$$\begin{aligned} &\bar{t} = c_0 t/R_0 \qquad \bar{z} = z/R_0 \qquad \bar{P} = PR_0/(2\mu H) \\ &\bar{v} = v/c_0 \qquad B = A/A_0 \qquad \kappa = \rho_s H/(\rho_f R_0) \end{aligned} \tag{2.4}$$

where c_0 is the Moens–Korteweg velocity, that is, $c_0^2 = 2\mu H/(\rho_f R_0)$, into the governing equations (2.1)–(2.3), and dropping the bars above letters for convenience, we obtain the following dimensionless equations:

$$\frac{\partial B}{\partial t} + \frac{\partial (Bv)}{\partial z} = 0 \tag{2.5}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + \frac{\partial P}{\partial z} = 0 \tag{2.6}$$

$$2BP - \kappa\left[\frac{\partial^2 B}{\partial t^2} - \frac{1}{2B}\left(\frac{\partial B}{\partial t}\right)^2\right] = \Phi(B) \tag{2.7}$$

where $\Phi(B)$ satisfies the condition $\Phi(1) = 0$. The asymptotic analysis in section 3 will be based on these dimensionless equations. Note that these equations involve two types of nonlinearities; one is due to the convective terms in equations (2.5) and (2.6) and the other results from the contribution of the nonlinearity introduced in the pressure–area relation in equation (2.7). It is easy to see that if one expands $\Phi(B)$ into a Taylor series about $B = 1$, and neglects nonlinear terms, equation (2.7) reduces to the following pressure–area relation:

$$2P - \kappa\frac{\partial^2 B}{\partial t^2} = \Phi'(1)(B-1) \tag{2.8}$$

which corresponds to the linear elastic case.

The function $\Phi(B)$ determines the behaviour of the solutions through the coefficient of the nonlinear term in the evolution equations. We now present $\Phi(B)$ corresponding to the strain energy functions of three different elastic materials, namely the Mooney–Rivlin elastic material and those proposed by Ishiara *et al* [3] and Demiray [4] (for details see [7]).

(i) For the Mooney–Rivlin elastic material the function $\Phi(B)$ is given as

$$\Phi(B) = B - B^{-1}. \tag{2.9}$$

(ii) For the strain energy function proposed by Ishiara *et al* [3] to represent certain elastomers, $\Phi(B)$ takes the following form:

$$\Phi(B) = (B - B^{-1})[1 + 2\beta(B + B^{-1} - 2)] \tag{2.10}$$

where β is a material constant.

(iii) For the strain energy function proposed by Demiray [4] for biological tissues, such as arterial wall, the function $\Phi(B)$ is obtained as

$$\Phi(B) = (B - B^{-1})\exp[\beta(B + B^{-1} - 2)] \tag{2.11}$$

where $\beta \geqslant 0$ is a material constant.

3. Nonlinear wave modulation near the marginal state

It is well known that the NLS equation arises as the governing equation for the amplitude in purely dispersive systems in which the linear dispersion equation does not admit complex frequencies for a real wavenumber. The nonlinear amplitude modulation in purely dispersive systems has been widely investigated in recent years using the reductive perturbation method formulated by Taniuti and Yajima [9], and other perturbation methods (see e.g. Jeffrey and Kawahara [5]). In a recent article [2], the amplitude modulation of periodic waves in fluid-filled distensible tubes was investigated using the reductive perturbation method.

Following the approach given in [5] the slow variables ξ and η are defined as

$$\xi = \epsilon(z - \lambda t) \qquad \eta = \epsilon^2 t \tag{3.1}$$

where λ is a real constant and ϵ is a small parameter measuring the weakness of the nonlinearity. Assuming that all the field variables have the following series solutions expanded in terms of the small parameter ϵ about a constant state:

$$\Theta = \sum_{n=0}^{\infty} \Theta_n \epsilon^n \qquad \Theta_n = \sum_{l=-\infty}^{\infty} \Theta_n^{(l)}(\xi, \eta) e^{il\theta} \tag{3.2}$$

where Θ is used to represent any of the field variables and $\theta = kz - \omega t$ is the phase function (where k is the wavenumber and ω is the frequency), then it is shown for a nonlinear visco-elastic tube that the amplitude modulation of waves is given by a dissipative NLS equation. In the absence of wall visco-elasticity, i.e. in the case of a nonlinear elastic tube, the dissipative NLS equation reduces to the following classical NLS equation:

$$i\frac{\partial B_1^{(1)}}{\partial \eta} + p\frac{\partial^2 B_1^{(1)}}{\partial \xi^2} + q|B_1^{(1)}|^2 B_1^{(1)} = 0 \tag{3.3}$$

where $B_1^{(1)}$ is the dimensionless internal area of the tube in the first order; the same equation is also valid, with slightly different coefficients, for the other first-order quantities, i.e. dimensionless fluid velocity $v_1^{(1)}$ and dimensionless pressure $P_1^{(1)}$.

The coefficients p and q in equation (3.3) are given as follows.

$$p = \frac{1}{2}\frac{d^2\omega}{dk^2} = -\frac{3\kappa\omega}{(2 + \kappa k^2)^2}$$

$$q = -\Big[3\Phi'''(B_0)B_0^2\kappa k^2(\kappa^2k^4 + 6\kappa k^2 + 12) - (\Phi''(B_0))^2 B_0^2 \frac{k^2}{\omega^2}(5\kappa^2k^4 + 18\kappa k^2 + 12)$$
$$+\Phi''(B_0)B_0(5\kappa^3k^6 + 34\kappa^2k^4 - 12\kappa k^2 - 48)$$
$$+\frac{\omega^2}{k^2}(\kappa^4k^8 - 38\kappa^3k^6 - 152\kappa^2k^4$$
$$-168\kappa k^2 - 48)\Big]\Big/[12B_0^2\omega\kappa(2+\kappa k^2)(\kappa^2k^4 + 6\kappa k^2 + 12)] \tag{3.4}$$

where a prime denotes differentiation with respect to the argument. Note that the coefficient p of the dispersive term is always negative for positive ω. If one considers the linear pressure–area relation in equation (2.8) rather than the nonlinear one in equation (2.7) for the case of $B_0 = 1$, the coefficient q would take the following form

$$q = \frac{\omega(6\kappa^3k^6 + 35\kappa^2k^4 + 78\kappa k^2 + 36)}{\kappa k^2(2+\kappa k^2)(\kappa^2k^4 + 6\kappa k^2 + 12)}. \tag{3.5}$$

Note that the coefficient q is always positive in the case of a linear elastic tube.

It is well known that the NLS equation has a plane-wave solution of a constant amplitude and that the modulational instability of this plane wave is determined by the sign of the product of the coefficients of the nonlinear and dispersive terms. Therefore, the marginal state of the modulational instability is given by the condition that the coefficient of the nonlinear term or the coefficient of the dispersive term vanishes. Since the coefficient p of the dispersive term given in $(3.4)_1$ is always negative for both linear and nonlinear elastic materials, the marginal state of instability of plane waves is given by the critical wavenumbers for which the coefficient of the nonlinear term vanishes. As is shown numerically in [2] there are no critical wavenumbers for linear elastic and Mooney–Rivlin elastic materials. However, depending on the initial deformation of the tube wall, i.e. depending on the value of B_0, the presence of such critical wavenumbers is observed for the materials proposed by Ishiara *et al* and Demiray in the same study. The variation of the critical wavenumbers k (for which q becomes zero) with initial deformation B_0 is presented in figure 1. The aim of this paper is to investigate the behaviour of the field equations near these critical points, i.e. near the marginal state. Since the coefficient q of the nonlinear term vanishes for the critical wavenumbers, the NLS equation derived and the asymptotic expansion method used are no longer valid near these values. Therefore, a new ordering should be used to be able to include nonlinearity in the analysis. This can be done by intensifying the effect of nonlinearity in the expansion.

We now examine the amplitude modulation of periodic waves near the marginal state in a fluid-filled nonlinear elastic tube with dimensionless governing equations given by (2.5)–(2.7). Here the order of nonlinearity is assumed to be $\mathcal{O}(\epsilon^{1/2})$ instead of $\mathcal{O}(\epsilon)$ which is the case for the NLS equation. We shall derive a new governing equation near the marginal state at which the wavenumber of the carrier wave takes the critical value k, that is, the evolution equation to be found in what follows will be valid in the vicinity of these critical points.

First, in order to see the dispersive character of the one-dimensional model, *equations (2.5)–(2.7) are linearized about a constant state:* $B = B_0$, $P = P_0$ *and* $v = 0$. *In* this case B_0 and P_0 have to satisfy the relation

$$2B_0P_0 = \Phi(B_0) \tag{3.6}$$

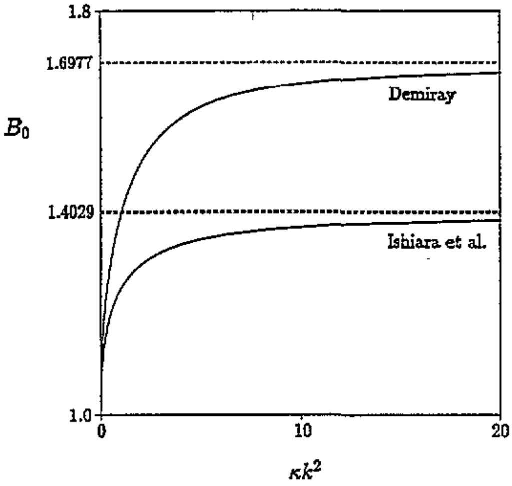

Figure 1. The variation of critical wavenumber k with B_0, for the strain energy functions proposed by Ishiara *et al* and Demiray.

as a consequence of equation (2.7). Now assume harmonic wave solutions for the linearized equations that are in the form $\{\hat{B}, \hat{v}, \hat{P}\} \exp[\mathrm{i}(kz-\omega t)]$ where k and ω respectively represent wavenumber and frequency, and $\hat{B}$, $\hat{P}$ and $\hat{v}$ are the amplitudes of 'small' perturbations. Substitution for these solutions in the linearized equations leads to a set of homogeneous equations in terms of these amplitudes and, for non-trivial solutions, the determinant of the system must vanish. This condition yields the following dispersion relation:

$$\omega = k\left[\frac{\Phi'(B_0) - 2P_0}{2+\kappa k^2}\right]^{1/2} \tag{3.7}$$

where ω is a real quantity. Since a real wave speed is assumed, the inequalities $\Phi'(B_0) - 2P_0 \geqslant 0$ when $B_0 \neq 1$ and $\Phi'(1) \geqslant 0$ when $B_0 = 1$ should hold.

We now consider the wavenumbers near the critical wavenumber k where the coefficient of the nonlinear term of the NLS equation vanishes. To be able to take into account the nonlinearity, all the field variables, i.e. B, v and P, are assumed to have the following series solutions expanded in terms of the small parameter $\epsilon^{1/2}$ about a constant state:

$$\Theta = \sum_{n=0}^{\infty} \Theta_n (\epsilon^{1/2})^n \qquad \Theta_n = \sum_{l=-\infty}^{\infty} \Theta_n^{(l)}(\xi, \eta) \mathrm{e}^{\mathrm{i}l\theta} \tag{3.8}$$

where Θ is used to represent any of these field variables in which $\Theta_0 = \{B_0, 0, P_0\}$, and $\theta = kz - \omega t$ is the phase function. We have also assumed that the reality conditions $\Theta_n^{(l)} = \Theta_n^{(-l)*}$ (the asterisk denotes complex conjugation) hold so that Θ_n are real. As can be seen from the expansion (3.8), by assuming that the amplitude functions $\Theta_n^{(l)}$ depend on the slow variables ξ and η, and that the phase function θ depends on the fast variables z and t, we have decomposed the solution into a rapidly varying one associated with the oscillations and a slowly varying envelope of a carrier wave with these fast oscillations.

As we are dealing with the nonlinear self-modulation of pressure waves centred around the frequency ω and the critical wavenumber k, the coefficients of higher harmonics, except those of first harmonics, are taken as zero for the first-order quantities, i.e. $\Theta_1^{(l)} = 0$ for $l \neq \pm 1$. In this case the first-order quantities take the following form:

$$\Theta_1 = \Theta_1^{(1)}(\xi, \eta)\mathrm{e}^{\mathrm{i}\theta} + \Theta_1^{(-1)}(\xi, \eta)\mathrm{e}^{-\mathrm{i}\theta}. \tag{3.9}$$

We now want to discover how the slowly varying amplitudes $\Theta_1^{(1)}$ are affected by nonlinearity near the marginal state. Furthermore, in order to eliminate the self-resonance we require that the dispersion relation (3.7) is not satisfied by the pairs (lk, lw) for $l \geqslant 2$ but is satisfied by the pair (k, ω). This can be expressed in the form $D(lk, l\omega) \neq 0$ for $l \geqslant 2$ where $D(lk, l\omega)$ is defined as

$$D(lk, l\omega) = l^2\omega^2(2 + \kappa l^2 k^2) - l^2k^2[\Phi'(B_0) - 2P_0]. \tag{3.10}$$

Note that this expression gives the linear dispersion relation (3.7) for $l = 1$.

Now, substituting the expansion (3.8) together with the coordinate stretching equation (3.1) into the field equations (2.5)–(2.7), and equating the terms with the same powers of $\epsilon^{1/2}$, we obtain a hierarchy of perturbation equations; since the elements of the hierarchy are complicated, only the results obtained for each order will be presented here.

For $\mathcal{O}(\epsilon^{1/2})$, recalling that $B_1^{(l)} = v_1^{(l)} = P_1^{(l)} = 0$ for $l \neq \pm 1$ and using the condition $D(k, \omega) = 0$, we find the following results:

$$v_1^{(1)} = \nu_1 B_1^{(1)} \qquad P_1^{(1)} = \pi_1 B_1^{(1)} \qquad B_1^{(1)} \text{ arbitrary} \tag{3.11}$$

where

$$\nu_1 = \frac{\omega}{B_0 k} \qquad \pi_1 = \frac{\omega^2}{B_0 k^2}.$$

For brevity, the arbitrary function $B_1^{(1)}$ will be represented by ϕ.

For $\mathcal{O}(\epsilon)$, setting the coefficients of each mode equal to zero and using the results obtained for $\mathcal{O}(\epsilon^{1/2})$, we obtain

$$\begin{array}{ll}
l = 0 & (2P_0 - \Phi')B_2^{(0)} + 2B_0 P_2^{(0)} = [\Phi'' - (4 + \kappa k^2)\pi_1]|\phi|^2 \\
l = 1 & v_2^{(1)} = \nu_1 B_2^{(1)} \qquad P_2^{(1)} = \pi_1 B_2^{(1)} \qquad B_2^{(1)} \text{ arbitrary} \\
l = 2 & B_2^{(2)} = \beta_2 \phi^2 \qquad v_2^{(2)} = \nu_2 \phi^2 \qquad P_2^{(2)} = \pi_2 \phi^2 \\
l \geqslant 3 & v_2^{(l)} = B_2^{(l)} = P_2^{(l)} = 0
\end{array} \tag{3.12}$$

where the coefficients β_2, ν_2 and π_2 are functions of the wavenumber k and of Φ; their explicit forms are given by (A.1) in appendix A. For convenience, the argument of Φ will not be written explicitly. Redefining the function $B_1^{(1)}$ as

$$B_1^{(1)} = B_1^{(1)} + \epsilon^{1/2} B_2^{(1)} \tag{3.13}$$

the arbitrary function $B_2^{(1)}$ can be included in the function $B_1^{(1)}$. Thus, the arbitrary function $B_2^{(1)}$ can be taken as zero. In this case $B_2^{(1)} = v_2^{(1)} = P_2^{(1)} = 0$.

For $\mathcal{O}(\epsilon^{3/2})$, using the previously obtained results, we find the following results for each mode:

$$\begin{array}{ll}
l = 0 & (2P_0 - \Phi')B_3^{(0)} + 2B_0 P_3^{(0)} = 0 \\
l = 1 & D(k, \omega)B_3^{(1)} + \left(\lambda - \dfrac{d\omega}{dk}\right)\dfrac{\partial\phi}{\partial\xi} + q|\phi|^2\phi = 0 \\
 & v_3^{(1)} = \nu_1 B_3^{(1)} + \dfrac{\nu_1}{i\omega}\left(\lambda - \dfrac{\omega}{k}\right)\dfrac{\partial\phi}{\partial\xi} + v_1|\phi|^2\phi \\
 & P_3^{(1)} = \pi_1 B_3^{(1)} + \dfrac{2\nu_1}{ik}\left(\lambda - \dfrac{\omega}{k}\right)\dfrac{\partial\phi}{\partial\xi} + p_1|\phi|^2\phi \\
l = 2 & B_3^{(2)} = v_3^{(2)} = P_3^{(2)} = 0 \\
l = 3 & B_3^{(3)} = \beta_3\phi^3 \qquad v_3^{(3)} = \nu_3\phi^3 \qquad P_3^{(3)} = \pi_3\phi^3 \\
l \geqslant 4 & v_3^{(l)} = B_3^{(l)} = P_3^{(l)} = 0
\end{array} \tag{3.14}$$

where the coefficient q is given in equation $(3.4)_2$. The explicit forms of the coefficients ν_1, p_1, β_3, ν_3 and π_3 in equations (3.14) are given by (A.2) and (A.3) in appendix A. Since the coefficient q of the nonlinear term of the NLS equation is zero and $D(k, \omega) = 0$, the function $B_3^{(1)}$ is found to be an arbitrary function. In such a case the non-triviality condition for ϕ yields $\lambda = \mathrm{d}\omega/\mathrm{d}k$; that is, λ introduced by the coordinate stretching equation (3.1) is the group velocity of the carrier wave. Similarly, the arbitrary function $B_3^{(1)}$ can be included in $B_1^{(1)}$ by suitably redefining $B_1^{(1)}$; thus the function $B_3^{(1)}$ can be taken as zero and is removed from the expressions for $v_3^{(1)}$ and $P_3^{(1)}$.

For $\mathcal{O}(\epsilon^2)$, using the previously obtained results, we find the following equations for the zeroth mode:

$$
\begin{aligned}
&l = 0 \\
&-\lambda B_2^{(0)} + B_0 v_2^{(0)} = -2\nu_1 |\phi|^2 \\
&-\lambda v_2^{(0)} + P_2^{(0)} = -\frac{\pi_1}{B_0} |\phi|^2 \\
&2B_0 P_4^{(0)} + (2P_0 - \Phi')B_4^{(0)} - \frac{\mathrm{i}\nu_1}{k}\left[4\left(\lambda - \frac{\omega}{k}\right) + \kappa\lambda k^2\right]\left(\phi^* \frac{\partial\phi}{\partial\xi} - \phi\frac{\partial\phi^*}{\partial\xi}\right) + \rho|\phi|^4 = 0
\end{aligned}
\tag{3.15}
$$

where the coefficient ρ is given by (A.5) in appendix A. Solving the equations given in $(3.15)_{1,2}$ and $(3.12)_1$ together, the functions $B_2^{(0)}$, $v_2^{(0)}$ and $P_2^{(0)}$ are found as

$$
B_2^{(0)} = \beta_0 |\phi|^2 \qquad v_2^{(0)} = \nu_0 |\phi|^2 \qquad P_2^{(0)} = \pi_0 |\phi|^2 \tag{3.16}
$$

where β_0, ν_0 and π_0 are functions of k and Φ and their explicit forms are given by (A.4) in appendix A. For the first mode of $\mathcal{O}(\epsilon^2)$

$$
l = 1 \qquad v_4^{(1)} = \nu_1 B_4^{(1)} \qquad P_4^{(1)} = \pi_1 B_4^{(1)} \qquad B_4^{(1)} \text{ arbitrary.}
$$

Similarly, by suitably redefining the arbitrary function $B_1^{(1)}$, the function $B_4^{(1)}$ may be included in $B_1^{(1)}$ and, thus, $B_4^{(1)}$ is taken as zero. In this case $B_4^{(1)} = v_4^{(1)} = P_4^{(1)} = 0$. For the second and third modes of $\mathcal{O}(\epsilon^2)$ the following results are found:

$$
\begin{aligned}
l = 2 \qquad & B_4^{(2)} = \mathrm{i}\alpha_1 \frac{\partial\phi^2}{\partial\xi} + \alpha_2 |\phi|^2\phi^2 \\
& v_4^{(2)} = \mathrm{i}\gamma_1 \frac{\partial\phi^2}{\partial\xi} + \gamma_2 |\phi|^2\phi^2 \\
& P_4^{(2)} = \mathrm{i}\delta_1 \frac{\partial\phi^2}{\partial\xi} + \delta_2 |\phi|^2\phi^2 \\
l = 3 \qquad & B_4^{(3)} = v_4^{(3)} = P_4^{(3)} = 0.
\end{aligned}
\tag{3.17}
$$

The explicit forms of the coefficients α_1, α_2, γ_1, γ_2, δ_1 and δ_2 are listed in (A.6), (A.7) and (A.8) in appendix A. Since we do not need functions obtained from the higher order modes of $\mathcal{O}(\epsilon^2)$ they will not be given here.

For $\mathcal{O}(\epsilon^{5/2})$, using the previously obtained results, we find the following equations for the zeroth-order $l = 0$:

$$
-\lambda B_3^{(0)} + B_0 v_3^{(0)} = 0 \qquad -\lambda v_3^{(0)} + P_3^{(0)} = 0. \tag{3.18}
$$

If equations (3.18) are solved together with the equation given in $(3.14)_1$, the following result is obtained:

$$
B_3^{(0)} = v_3^{(0)} = P_3^{(0)} = 0. \tag{3.19}
$$

Recalling equation (3.15), we need two more conditions to calculate the functions $B_4^{(0)}$, $v_4^{(0)}$ and $P_4^{(0)}$. These conditions are obtained from equations (2.6) and (2.7) for the zeroth mode ($l = 0$) of ϵ^3:

$$-\lambda\frac{\partial B_4^{(0)}}{\partial\xi} + \frac{\partial B_2^{(0)}}{\partial\eta} + B_0\frac{\partial v_4^{(0)}}{\partial\xi} + \frac{\partial}{\partial\xi}(B_1^{(1)}v_3^{(-1)} + B_1^{(-1)}v_3^{(1)} + B_2^{(-2)}v_2^{(2)} + B_2^{(0)}v_2^{(0)} + B_2^{(2)}v_2^{(-2)}) = 0$$

$$-\lambda\frac{\partial v_4^{(0)}}{\partial\xi} + \frac{\partial v_2^{(0)}}{\partial\eta} + \frac{\partial P_4^{(0)}}{\partial\xi} + \frac{\partial}{\partial\xi}(v_1^{(1)}v_3^{(-1)} + v_1^{(-1)}v_3^{(1)} + v_2^{(-2)}v_2^{(2)} + \tfrac{1}{2}v_2^{(0)}v_2^{(0)}) = 0. \tag{3.20}$$

If equations (3.20) are solved together with that given in $(3.15)_3$ (for $l = 0$ of ϵ^2) the following results are obtained:

$$B_4^{(0)} = \mathrm{i}r_1\left(\phi\frac{\partial\phi^*}{\partial\xi} - \phi^*\frac{\partial\phi}{\partial\xi}\right) + r_2|\phi|^4 + r_3\int\frac{\partial|\phi|^2}{\partial\eta}\,\mathrm{d}\xi$$

$$v_4^{(0)} = \mathrm{i}u_1\left(\phi\frac{\partial\phi^*}{\partial\xi} - \phi^*\frac{\partial\phi}{\partial\xi}\right) + u_2|\phi|^4 + u_3\int\frac{\partial|\phi|^2}{\partial\eta}\,\mathrm{d}\xi$$

$$P_4^{(0)} = \mathrm{i}s_1\left(\phi\frac{\partial\phi^*}{\partial\xi} - \phi^*\frac{\partial\phi}{\partial\xi}\right) + s_2|\phi|^4 + s_3\int\frac{\partial|\phi|^2}{\partial\eta}\,\mathrm{d}\xi. \tag{3.21}$$

The explicit forms of the coefficients r_1, r_2, r_3, u_1, u_2, u_3, s_1, s_2 and s_3 are given in (A.9), (A.10) and (A.11) in appendix A. For the first mode ($l = 1$) of $\mathcal{O}(\epsilon^{5/2})$, after some tedious calculations the following nonlinear integro-differential equation is obtained as a compatibility condition:

$$\mathrm{i}\frac{\partial\phi}{\partial\eta} + p\frac{\partial^2\phi}{\partial\xi^2} + \sigma_1|\phi|^4\phi + \mathrm{i}\sigma_2|\phi|^2\frac{\partial\phi}{\partial\xi} + \mathrm{i}\sigma_3\phi\frac{\partial|\phi|^2}{\partial\xi} + \sigma_4\phi\int\frac{\partial|\phi|^2}{\partial\eta}\,\mathrm{d}\xi = 0 \tag{3.22}$$

where the coefficients σ_1, σ_2, σ_3 and σ_4 are given by (B.1) in appendix B.

A similar evolution equation was derived by Johnson [10] in analysing the Stokes instability in fluid flows near the critical values of $kh = 1.363$. As pointed out by Kakutani and Michihiro [11], this integro-differential equation can be reduced to a differential equation by simply removing the integral part of the equation. If equation (3.22) is multiplied by ϕ^* and subtracted from the complex conjugate of the same equation the last term in the evolution equation takes the following form:

$$\int\frac{\partial|\phi|^2}{\partial\eta}\,\mathrm{d}\xi = \mathrm{i}p\left(\phi^*\frac{\partial\phi}{\partial\xi} - \phi\frac{\partial\phi^*}{\partial\xi}\right) + \frac{\sigma_2 + 2\sigma_3}{2}|\phi|^4. \tag{3.23}$$

If expression (3.23) is introduced into equation (3.22), the final form of the evolution equation is

$$\mathrm{i}\frac{\partial\phi}{\partial\eta} + p\frac{\partial^2\phi}{\partial\xi^2} + q_1|\phi|^4\phi + \mathrm{i}q_2|\phi|^2\frac{\partial\phi}{\partial\xi} + \mathrm{i}q_3\phi\frac{\partial|\phi|^2}{\partial\xi} = 0 \tag{3.24}$$

where

$$q_1 = \sigma_1 + \sigma_4(\sigma_2 + 2\sigma_3)/2 \qquad q_2 = \sigma_2 + 2p\sigma_4 \qquad q_3 = \sigma_3 - p\sigma_4.$$

The evolution equation (3.24) is called the generalized nonlinear Schrödinger (GNLS) equation. As a result of equations (3.11) the same equation is valid with slightly different coefficients for $v_1^{(1)}$ and $P_1^{(1)}$. The GNLS equation arises in a wide variety of fields as an equation describing the self-modulation of the one-dimensional monochromatic plane waves near the marginal state in dispersive media. This higher-order evolution equation has already been obtained by Kakutani and Michihiro [11] for gravity water waves. The same equation has also been derived formally by Parkes in [12] for a general dispersive system involving a single dependent variable.

As special cases, the GNLS equation contains the following equations:

$$\mathrm{i}\frac{\partial\phi}{\partial\eta} + p\frac{\partial^2\phi}{\partial\xi^2} + \mathrm{i}q_2\frac{\partial}{\partial\xi}(|\phi|^2\phi) = 0 \tag{3.25}$$

in the case of $q_1 = 0$ and $q_2 = q_3$, and

$$\mathrm{i}\frac{\partial\phi}{\partial\eta} + p\frac{\partial^2\phi}{\partial\xi^2} + \mathrm{i}q_2|\phi|^2\frac{\partial\phi}{\partial\xi} = 0 \tag{3.26}$$

in the case of $q_1 = 0$ and $q_3 = 0$. The first equation is called the derivative nonlinear Schrödinger (DNLS) equation; the DNLS equation governs the propagation of nonlinear Alfven waves in plasma [13]. The second equation describes the self-modulation of the complex amplitude of solutions to the Benjamin–Ono equation and is derived in [14].

The complete integrability of the GNLS equation given in (3.24) has been studied by Clarkson and Cosgrove [15] via a Painlevé analysis. It was shown that the GNLS equation posseses the Painlevé property in the sense of Weiss *et al* [16] if the following condition holds between the coefficients of the GNLS equation:

$$4pq_1 = q_3(q_3 - q_2). \tag{3.27}$$

In our case the relation (3.27) gives a constraint between the wavenumber k, the frequency ω and the function $\Phi(B)$. If $q_2 = q_3$ then $q_1 = 0$ and the GNLS equation (3.24) becomes the DNLS equation which is known to be completely integrable [13].

4. Exact solutions of the GNLS equation

Kundu [17] has used a gauge transformation to transform equation (3.22) to the DNLS equation from which exact solutions to the GNLS equation are found. Pathria and Morris [8] have used a similar transformation which is a generalization of that given in [17]. Some other exact solutions to the GNLS equation are given by Florjanczyk and Gagnon [18] using the symmetry reduction method. New dimensional reductions and exact solutions for the GNLS equation are also given by Clarkson [19] using an extension of the direct method originally developed by Clarkson and Kruskal [20].

Redefining the time variable η as $\tau = p\eta$, the GNLS equation takes the following form:

$$\mathrm{i}\frac{\partial\phi}{\partial\tau} + \frac{\partial^2\phi}{\partial\xi^2} + q_1|\phi|^4\phi + \mathrm{i}q_2|\phi|^2\frac{\partial\phi}{\partial\xi} + \mathrm{i}q_3\phi\frac{\partial|\phi|^2}{\partial\xi} = 0 \tag{4.1}$$

where all the coefficients are divided by the coefficient p. We now consider the following solutions to equation (4.1) ([8]):

$$\phi(\xi,\tau) = f(\zeta)\exp\{\mathrm{i}[\vartheta(\xi,\tau) + g(\xi,\tau)]\} \tag{4.2}$$

where $\zeta = \xi - c\tau$ and $g(\xi,\tau)$ and $\vartheta(\xi,\tau)$ are real functions defined by

$$g(\xi,\tau) = c(\xi - b\tau)/2 + d \qquad \vartheta(\xi,\tau) = 2\delta\int f^2(\zeta)\,\mathrm{d}\zeta. \tag{4.3}$$

Here b, c and d are arbitrary constants and δ is given as

$$\delta = -\tfrac{1}{8}(2q_3 + q_2).$$

Moreover, the function f satisfies the following ordinary differential equation:

$$z'^2 = -\frac{4\overline{q}_1}{3}z^4 + c\overline{q}_2 z^3 - (2bc - c^2)z^2 \tag{4.4}$$

where the definition $z = f^2$ is used and a prime denotes differentiation with respect to the argument. The coefficients seen in equation (4.4) are defined as

$$\overline{q}_1 = q_1 + 4\delta^2 + 2\delta q_3 - \delta q_2 \qquad \overline{q}_2 = q_2. \tag{4.5}$$

The roots of the polynomial on the right-hand side of equation (4.4) are real if the following condition is satisfied:

$$1 + \frac{16\overline{q}_1}{3\overline{q}_2^2}\left(1 - 2\frac{b}{c}\right) > 0. \tag{4.6}$$

The coefficients $\overline{q}_1$ and $\overline{q}_2$ given in equation (4.5) depend on the function $\Phi(B)$ and the wavenumber k. If the variations of the coefficients $\overline{q}_1$ and $\overline{q}_2$ with wavenumber k are plotted (figure 2) for the material proposed by Ishiara *et al* (the figure is almost the same as the one corresponding to the material proposed by Demiray), it is observed that $\overline{q}_1$ has both positive and negative values whereas $\overline{q}_2$ is always negative. Since b and c are arbitrary constants, the restriction (4.6) is always fulfilled. In such a case it is reasonable to look for solutions for both positive and negative values of $\overline{q}_1$.

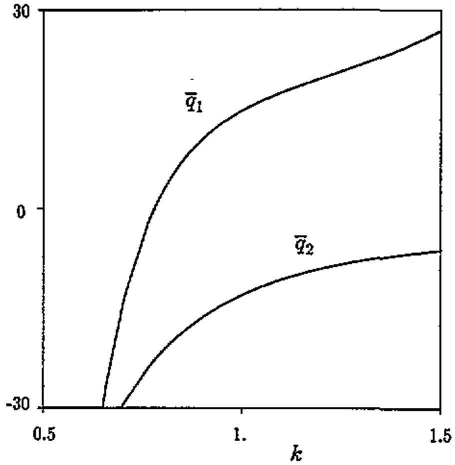

Figure 2. The variation of coefficients $\overline{q}_1$ and $\overline{q}_2$ with wavenumber k for the strain energy function proposed by Ishiara *et al* ($\kappa = 1$, $\beta = 1$).

If $\overline{q}_1 < 0$, solving equation (4.4) and using equation $(4.3)_2$ the following solitary-wave solution to equation (4.1) is obtained:

$$\begin{aligned} f(\xi, \tau) &= \left[\frac{z_1 z_2}{z_1 + (z_1 - z_2)\sinh^2\chi}\right]^{1/2} \\ \vartheta(\xi, \tau) &= 2\delta\left(\frac{-3}{\overline{q}_1}\right)^{1/2} \operatorname{arctanh}\left[\left(\frac{z_2}{z_1}\right)^{1/2} \tanh\chi\right] \\ \chi &= \left(\frac{-z_1 z_2 \overline{q}_1}{3}\right)^{1/2}(\xi - c\tau) + e \end{aligned} \tag{4.7}$$

where z_1 and z_2 are the positive roots of the fourth degree polynomial given in the right-hand side of equation (4.4) (if expression (4.6) is less than unity for positive c it is possible to find two positive roots) and e is an arbitrary constant.

If $\overline{q}_1 > 0$ and z_1 and z_2 are the positive roots of the polynomial given in equation (4.4) (if expression (4.6) is less than unity for negative c it is possible to find two positive roots) the following solitary-wave solution to equation (4.1) is found:

$$\begin{aligned} f(\xi,\tau) &= \left[\frac{z_1 z_2}{z_1 + (z_2 - z_1)\cos^2\chi}\right]^{1/2} \\ \vartheta(\xi,\tau) &= 2\delta\left(\frac{3}{\overline{q}_1}\right)^{1/2} \arctan\left[\left(\frac{z_1}{z_2}\right)^{1/2} \tan\chi\right] \\ \chi &= \left(\frac{z_1 z_2 \overline{q}_1}{3}\right)^{1/2} (\xi - c\tau) + e. \end{aligned} \tag{4.8}$$

If $\overline{q}_1 > 0$, z_1 is a positive root and z_2 is a negative root of the polynomial given in equation (4.4) (if the expression (4.6) is greater than unity for negative c it is possible to find a positive root and a negative root) the solitary-wave solution to equation (4.1) is found:

$$\begin{aligned} f(\xi,\tau) &= \left[\frac{z_1 z_2}{z_2 + (z_2 - z_1)\sinh^2\chi}\right]^{1/2} \\ \vartheta(\xi,\tau) &= 2\delta\left(\frac{3}{\overline{q}_1}\right)^{1/2} \arctan\left[\left(\frac{-z_1}{z_2}\right)^{1/2} \tanh\chi\right] \\ \chi &= \left(\frac{-z_1 z_2 \overline{q}_1}{3}\right)^{1/2} (\xi - c\tau) + e. \end{aligned} \tag{4.9}$$

It should be noted that the GNLS equation has solitary-wave solutions for both positive and negative values of $\overline{q}_1$.

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Appendix A.

The explicit forms of the coefficients β_2, ν_2 and π_2 are given as

$$\begin{aligned} \beta_2 &= \frac{1}{6\kappa\omega^2}[\Phi'' + (2 + \kappa k^2)\pi_1] \\ \nu_2 &= \frac{1}{6B_0\kappa\omega k}[\Phi'' + (2 - 5\kappa k^2)\pi_1] \\ \pi_2 &= \frac{1}{6B_0\kappa k^2}[\Phi'' + (2 - 8\kappa k^2)\pi_1]. \end{aligned} \tag{A.1}$$

The explicit forms of the coefficients v_1 and p_1 are as follows

$$\begin{aligned} v_1 = -\frac{1}{B_0^2}\Bigg[&\frac{1}{3\kappa k\omega}(\Phi'' + (2 - 2\kappa k^2)\pi_1) \\ &+ \frac{1}{2P_0 - \Phi' + 2\lambda^2}\left(\frac{\omega}{k} + \lambda\right)(4\lambda\nu_1 - (2 + \kappa k^2)\pi_1 + \Phi'') - 2\nu_1\Bigg] \end{aligned}$$

$$p_1 = -\frac{\nu_1}{B_0}\left[\frac{1}{2\kappa k\omega}(\Phi'' + (2-3\kappa k^2)\pi_1) + \frac{1}{2P_0 - \Phi' + 2\lambda^2}\left(\frac{\omega}{k} + 2\lambda\right)(4\lambda\nu_1 - (2+\kappa k^2)\pi_1 + \Phi'') - 4\nu_1\right]. \quad (A.2)$$

The coefficients β_3, ν_3 and π_3 are

$$\beta_3 = \frac{9k^2}{D(3k,3\omega)}\left[\frac{2}{3B_0\kappa k^2}(\Phi'' + (1-\kappa k^2)\pi_1) + \frac{1}{6B_0}(3\Phi'' + (4-\kappa k^2)\pi_1) + \tfrac{1}{6}\Phi''' + \frac{1}{6\kappa\omega^2}(\Phi'')^2\right]$$
$$\nu_3 = \nu_1\beta_3 - \frac{1}{3B_0^2\kappa k\omega}[\Phi'' + (2-2\kappa k^2)\pi_1] \quad (A.3)$$
$$\pi_3 = \pi_1\beta_3 - \frac{1}{2B_0^2\kappa k^2}[\Phi'' + (2-3\kappa k^2)\pi_1].$$

The coefficients β_0, ν_0 and π_0 are

$$\beta_0 = \frac{1}{2P_0 - \Phi' + 2\lambda^2}[4\lambda\nu_1 - (2+\kappa k^2)\pi_1 + \Phi'']$$
$$\nu_0 = \frac{\lambda}{B_0}\beta_0 - \frac{2\nu_1}{B_0} \quad (A.4)$$
$$\pi_0 = \frac{\lambda^2}{B_0}\beta_0 - \frac{\nu_1}{B_0}(\frac{\omega}{k} + 2\lambda).$$

The coefficient ρ is given as

$$\rho = 4\beta_2\pi_2 + 4p_1 + 4\kappa k^2\pi_1\beta_2^2 + \frac{\kappa k^2\pi_1}{B_0}(1-\beta_0-3\beta_2) - \tfrac{1}{6}\Phi'''' - \Phi'''(\beta_2+\beta_0) - \tfrac{1}{2}\Phi''(2\beta_2^2 + \beta_0^2). \quad (A.5)$$

The coefficients α_1 and α_2 are

$$\alpha_1 = \left[\lambda\omega(1+4\kappa k^2)\beta_2 + B_0k\left(\lambda - \frac{\omega}{k}\right)\nu_2 - B_0k\pi_2 - \tfrac{3}{2}k\pi_1 - \tfrac{1}{2}\lambda\kappa k^3\nu_1\right]/D(2k,2\omega)$$
$$\alpha_2 = \left[4\omega v_1k - 2p_1k^2 + 4\omega\nu_3k + 2\omega\beta_0\nu_2k + 2\omega\nu_0\beta_2k + 2B_0\nu_0\nu_2k^2 - 2(\pi_3 + \beta_0\pi_2 + \pi_0\beta_2)k^2 - 3\kappa k^4\pi_1\beta_3 + \frac{\kappa k^4\pi_1}{2B_0}(2\beta_2 - \beta_0) + \frac{k^2}{6}\Phi'''' + \frac{k^2}{2}\Phi'''(2\beta_2+\beta_0) + k^2\Phi''(\beta_2\beta_0 + \beta_3)\right]/D(2k,2\omega). \quad (A.6)$$

The coefficients γ_1 and γ_2 are

$$\gamma_1 = \nu_1\alpha_1 - \frac{\lambda\beta_2\nu_1}{2\omega} + \frac{\nu_2}{2k} + \frac{\nu_1^2}{2\omega} + \frac{\nu_1}{2B_0\omega}\left(\lambda - \frac{\omega}{k}\right)$$
$$\gamma_2 = \nu_1\alpha_2 - \frac{1}{B_0}(v_1 + \nu_3 + \beta_0\nu_2 + \beta_2\nu_0 + \beta_3\nu_1). \quad (A.7)$$

The coefficients δ_1 and δ_2 are

$$\delta_1 = \frac{\omega}{k}\gamma_1 - \frac{\lambda\nu_2}{2k} + \frac{\pi_2}{2k} + \frac{\pi_1\nu_1}{4\omega} + \frac{\nu_1^2}{2\omega}\left(\lambda - \frac{\omega}{k}\right)$$
$$\delta_2 = \frac{\omega}{k}\gamma_2 - \nu_1\nu_3 - \nu_1 v_1 - \nu_0\nu_2. \quad (A.8)$$

The coefficients r_1, r_2 and r_3 are given as

$$r_1 = -\frac{4\kappa\omega\nu_1}{(2+\kappa k^2)^2(2P_0-\Phi'+2\lambda^2)}$$
$$r_2 = \frac{1}{2P_0-\Phi'+2\lambda^2}[\lambda(4v_1+2\beta_0\nu_0+4\beta_2\nu_2)+B_0(4\nu_1 v_1+2\nu_2^2+\nu_0^2)-\rho] \qquad \text{(A.9)}$$
$$r_3 = \frac{2B_0}{2P_0-\Phi'+2\lambda^2}\left(\frac{\lambda}{B_0}\beta_0+\nu_0\right).$$

The coefficients u_1, u_2 and u_3 are given as

$$u_1 = \frac{\lambda r_1}{B_0} - \frac{1}{B_0^2 k}\left(\lambda-\frac{\omega}{k}\right)$$
$$u_2 = \frac{\lambda r_2}{B_0} - \frac{1}{B_0}(2\beta_2\nu_2+\beta_0\nu_0+2v_1) \qquad \text{(A.10)}$$
$$u_3 = \frac{\lambda r_3}{B_0} - \frac{\beta_0}{B_0}.$$

The coefficients s_1, s_2 and s_3 are given as

$$s_1 = \frac{\lambda^2 r_1}{B_0} - \frac{1}{B_0^2 k}\left(\lambda-\frac{\omega}{k}\right)\left(\lambda+\frac{\omega}{k}\right)$$
$$s_2 = \frac{\lambda^2 r_2}{B_0} - \frac{\lambda}{B_0}(2v_1+\beta_0\nu_0+2\beta_2\nu_2) - 2v_1\nu_1 - \nu_2^2 - \tfrac{1}{2}\nu_0^2 \qquad \text{(A.11)}$$
$$s_3 = \frac{\lambda^2 r_3}{B_0} - \left(\frac{\lambda}{B_0}\beta_0+\nu_0\right).$$

Appendix B.

The coefficients of the evolution equation (3.22), σ_1, σ_2, σ_3 and σ_4 are given as

$$\begin{aligned}
\sigma_1 = \frac{k^2}{2\omega(2+\kappa k^2)}\Big[&2(p_1-B_0v_1\nu_1)(\beta_0+\beta_2)-2B_0v_1(\nu_0+\nu_2)-4B_0\nu_1(u_2+\gamma_2)\\
&-2B_0\nu_1(\beta_2\nu_3+\beta_3\nu_2)+2(s_2+\delta_2)-2B_0\nu_2\nu_3+2(\beta_2\pi_3+\beta_3\pi_2)\\
&+2\kappa k^2\pi_1(\alpha_2+3\beta_2\beta_3)-\frac{\kappa k^2\pi_1}{B_0}\left(4\beta_2^2+2\beta_0\beta_2-4\beta_2+\tfrac{5}{2}\beta_3-\frac{\beta_0}{B_0}+\frac{1}{B_0^2}\right)\\
&-\tfrac{1}{2}\Phi''''-\tfrac{1}{6}\Phi''''(4\beta_2+3\beta_0)-\tfrac{1}{2}\Phi'''(2\beta_2^2+\beta_0^2+2\beta_0\beta_2+\beta_3)\\
&-\Phi''\beta_2\beta_3-\Phi''r_2-\Phi''\alpha_2\Big]
\end{aligned}$$
$$\begin{aligned}
\sigma_2 = \frac{k^2}{2\omega(2+\kappa k^2)}\Big[&\frac{4\nu_0}{k}\left(\lambda-\frac{\omega}{k}\right)+8B_0\nu_1(u_1-\gamma_1)-4(s_1-\delta_1)-4\nu_1\kappa k\lambda\beta_2\\
&+4\kappa k^2\pi_1\alpha_1+3\Phi''r_1-2\Phi''\alpha_1+\frac{4\nu_2}{k}\left(\lambda-\frac{\omega}{k}\right)+\frac{\kappa k\nu_1\lambda}{B_0}\Big]
\end{aligned}$$
$$\begin{aligned}
\sigma_3 = \frac{k^2}{2\omega(2+\kappa k^2)}\Big[&\frac{2}{k}\left(\lambda-\frac{\omega}{k}\right)(\nu_0+\beta_0\nu_1)-4B_0\nu_1u_1+\kappa k\lambda\nu_1\beta_0\\
&+4\pi_1\beta_2\left(\lambda-\frac{\omega}{k}\right)+2s_1+2k^2\nu_1\lambda\beta_2-\frac{\kappa k\nu_1\lambda}{B_0}-\Phi''r_1\Big]
\end{aligned}$$
$$\sigma_4 = -4B_0\nu_1u_3+2s_3-\Phi''r_3. \qquad \text{(B.1)}$$

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